

SEQUENCE OF INDUCED HAUSDORFF METRICS ON LIE GROUPS

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ABSTRACT. Let $\varphi : G \times (M, d) \rightarrow (M, d)$ be a left action of a Lie group on a differentiable manifold endowed with a metric d (distance function) compatible with the topology of M . Denote $gp := \varphi(g, p)$. Let X be a compact subset of M . Then the isotropy subgroup of X is a closed subgroup of G defined as $H_X := \{g \in G; gX = X\}$. The induced Hausdorff metric is a metric on the left coset manifold G/H_X defined as $d_X(gH_X, hH_X) = d_H(gX, hX)$, where d_H is the Hausdorff distance in M . Suppose that φ is transitive and that there exist $p \in M$ such that $H_X = H_p$. Then $gH_X \mapsto gp$ is a diffeomorphism that identifies G/H_X and M . In this work we define a discrete dynamical system on M . Let $d^1 = \hat{d}_X$, where \hat{d}_X stands for the intrinsic metric associated to d_X . We can iterate $\varphi : G \times (M \equiv G/H_X, d^1) \rightarrow (M \equiv G/H_X, d^1)$, in order to get d^2, d^3 and so on. We study the case where $M = G$, the left action $\varphi : G \times (G, d) \rightarrow (G, d)$ is the product of G , d is Lipschitz equivalent to a right invariant Riemannian metric on G and $X \ni e$ is a finite subset of G . We prove that in this case the sequence d^i converges pointwise to a metric d^∞ on M . Moreover, if the semigroup S_X generated by X is dense in G , then d^∞ is the distance function of a right invariant C^0 -Finsler metric.

1. INTRODUCTION

Before talking about the subject of this paper, we present two topics related to this work. We don't use the second topic here. It will only illustrate our main result (Theorem 3.11).

The first topic are intrinsic metrics on homogeneous spaces. Let G be a connected Lie group and H be a compact subgroup of G . Consider the canonical left action $\phi : G \times G/H \rightarrow G/H$ of G on the left coset manifold G/H . Let Δ be a G invariant completely nonholonomic distribution on G/H and F be a G -invariant norm on Δ . A horizontal curve is an absolutely continuous curve $\gamma : I \rightarrow G/H$ such that $\gamma'(t) \in \Delta$ whenever it is defined. Chow-Rashevskii theorem states that every pair of points in G can be connected by a horizontal curve (see [7], [17], [19]) and we proceed as in the definition of sub-Riemannian metric: The Carnot-Caratheodory-Finsler metric on G/H is given by

$$d_c(x, y) = \inf_{\gamma \in \mathcal{H}_{x,y}} \int_I F(\gamma(t), \gamma'(t)) dt,$$

where $\mathcal{H}_{x,y}$ is the set of horizontal curves connecting x and y (see [3, 4]).

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Berestovskii proved in Theorem 3 of [4] that if (M, d) is locally compact, locally contractible homogeneous space, with intrinsic metric d , then it is isometric to the quotient space G/H of a Lie group G over a compact subgroup H of it, endowed with a G -invariant Carnot-Caratheodory-Finsler metric. This result shows that if an intrinsic metric is homogeneous, then it has the tendency to gain extra regularity.

The second topic is the convergence of a one parameter family of Riemannian metrics to a “nice” Riemannian metric. Let us explain what is a nice Riemannian metric. For instance, every two-dimensional compact manifold admits a Riemannian metric with constant Gaussian curvature K , and the sign of K coincides with the sign of its Euler-Poincaré characteristic. Therefore Riemannian metrics with constant Gaussian curvatures are nice metrics. For higher dimensions, Riemannian manifolds with constant sectional curvature on M are locally isometric to a space form, and if we consider (M, \mathbf{g}) to be complete and simply connected, then it is a space form. Therefore, for general purposes, to study Riemannian metrics with constant sectional curvature is too restrictive although metrics with constant sectional curvature certainly are very nice. On the other hand, if M is a compact differentiable manifold with dimension $n \geq 3$, then there exist an infinite dimensional family of Riemannian metrics with constant scalar curvature. Therefore, for $n \geq 3$, metrics with constant scalar curvature can not be considered special. In the mid term, we have Einstein metrics, where the Ricci tensor is equal to the metric tensor times a constant. In particular, the covariant differential of the Ricci tensor is zero, and therefore they are Riemannian metrics with constant Ricci tensor. Einstein metrics are a special type of Riemannian metrics (see [5] for this subject).

If M is a compact n -dimensional differentiable manifold, then the Einstein metrics \mathbf{g} are the constant solutions $\mathbf{g}(t) \equiv \mathbf{g}$ of the normalized Ricci-Hamilton flow

$$\frac{\partial}{\partial t} \mathbf{g}(t) = -2\text{Ric}(t) + \frac{2}{n} R_{av} \mathbf{g}(t),$$

where Ric is the Ricci tensor and R_{av} is the average of the scalar curvature on M (see [8]). In some situations, a solution $(M, \mathbf{g}(t))$ of the normalized Ricci-Hamilton equation converges to an Einstein metric. One of the main result in this direction is in the Richard Hamilton’s seminal work [11], that states that the Ricci-Hamilton flow of a compact three dimensional Riemannian manifold with positive Ricci curvature converges to an Einstein metric, which is a nice metric. Moreover Einstein metrics in three dimensional manifolds has constant sectional curvature. This problem is widely known to be related to the solution of the Poincaré conjecture (see for instance [18]).

Now we present an outline of our work.

Let $\varphi : G \times M \rightarrow M$ be a left action of a Lie group on a differentiable manifold endowed with a metric d (distance function) compatible with the topology of M . As usual, we denote $gx := \varphi(g, x)$. Let X be a compact subset of M . The isotropy subgroup of X is a closed subgroup H_X of G given by

$$H_X = \{g \in G; gX = X\}.$$

Then there exist a unique differentiable structure on G/H_X , compatible with the quotient topology, such that the natural action $\phi : G \times G/H_X \rightarrow G/H_X$, given by $\phi(g, hH_X) = ghH_X$, is smooth. In [2] and [9] we define a metric d_X on G/H_X , which is called induced Haudorff metric, by

$$d_X(gH_X, hH_X) = d_H(gX, hX),$$

where d_H is the Hausdorff distance in (M, d) and gH_X denote the left coset of g in G/H_X . Suppose that φ is transitive and that there exist $p \in M$ such that $H_p = H_X$. Then the map $\eta : G/H_X \rightarrow M$ given by $gH_X \mapsto g.p$ is a diffeomorphism such that $\varphi(g, \eta(hH_X)) = \eta(\phi(g, hH_X))$, that is, φ can be identified with ϕ . We use the induced Hausdorff metric in order to create a discrete dynamical system of metrics on M . Define $d^1 = \hat{d}_X$, where the hat denotes the intrinsic metric associated to d_X . We can iterate this process in $\varphi : G \times (M, d^1) \rightarrow (M, d^1)$ in order to obtain d^2 , where $(G/H_X, d^1)$ is identified with (M, d^1) via η . Through this iteration, we can define a sequence of metrics d, d^1, d^2, \dots on M .

In this work we study the sequence of metrics d, d^1, d^2, \dots in the following case: M is G endowed with a metric that is Lipschitz equivalent to a right invariant Riemannian metric, $\varphi : G \times (G, d) \rightarrow (G, d)$ is the product of G and X is a finite subset of G containing the identity element e (It is implicit here that $H_X = \{e\}$). We prove that d^i it converges pointwise to a metric d^∞ . Moreover if the semigroup S_X generated by X is dense in G , then d^∞ is the distance function of a right invariant C^0 -Finsler metric. Therefore, if S_X is dense in G , this discrete dynamical system converges to a nice metric (right invariant metric) and its right invariance makes that it presents further regularity (it is C^0 -Finsler).

This work is divided as follows. In Section 2 we fix notations and give definitions and results that are necessary for this work. In Section 3 we study the properties of the sequence d, d^1, d^2, \dots . In Section 4, we prove that every connected Lie group G admits a finite subgroup $X \ni e$ such that S_X is dense in G .

2. PRELIMINARIES

In this section we give some definitions and results that are used in this work. They can be found in [6], [9], [12], [13], [14, 15] and [20]. For the sake of clearness, we usually don't give the definitions and results in their most general case.

Let G be a group and (M, d) be a metric space. Consider a left action G on M , $\varphi : G \times M \rightarrow M$. Then $ex = x$ for every $x \in M$ and $(gh)x = g(hx)$ for every $(g, h, x) \in G \times G \times M$. Every $\varphi_g := \varphi(g, \cdot)$ is a bijection. We say that a left action $\varphi : G \times M \rightarrow M$ is an action by isometries if every φ_g is an isometry. Analogously we say that φ is an action by homeomorphism if φ_g is a homeomorphism for every $g \in G$.

Remark 2.1. *If $\varphi : G \times M \rightarrow M$ is a left action of a group G on a metric space M , then it is immediate to see that $\phi : G \times M \rightarrow M$ defined as $\phi(g, x) = \varphi(g^{-1}, x)$ is a right action of G on M . Analogously it is immediate to see that if $\phi : G \times M \rightarrow M$ is a right action of G on M , then $\varphi(g, x) = \phi(g^{-1}, x)$ is a left action of G on M . Therefore, for some issues, it is indifferent to have a right action or a left action.*

Let (M, d) be a metric space. We denote the open ball with center p and radius r in (M, d) by $B_d(p, r)$. The topology induced by d is denoted by τ_d . The closure of a subset A in (M, d) is denoted by \bar{A} . When more than one metric or topology are involved, for instance we have metrics d and ρ and a topology τ on M , we use terms like d -neighborhood, τ -open subset, ρ -compact, etc.

Let A, B be compact subsets of (M, d) . The Hausdorff distance between A, B is given by

$$d_H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}.$$

It is well known that d_H is a metric on the family of compact subsets of M (see [6]).

Let $\varphi : G \times M \rightarrow M$ be a left action by homeomorphisms of a group G on a metric space (M, d) . If $X \subset M$ is a subset, then the isotropy subgroup of G with respect to X is defined by $H_X = \{g \in G; gX = X\}$ (see [2, 9]). Suppose that X is a compact subset in M . In Proposition 2.1 of [9], we define the induced Hausdorff metric on the quotient space G/H_X as $d_X(gH_X, hH_X) = d_H(gX, hX)$, where d_H is the Hausdorff distance in M and $gX = \{gx; x \in X\}$.

A partition \mathcal{P} of an interval $[a, b]$ is a subset $\{t_0, \dots, t_{n_{\mathcal{P}}}\} \subset [a, b]$ such that $a = t_0 < t_1 < \dots < t_{n_{\mathcal{P}}} = b$. The norm of \mathcal{P} is defined as $|\mathcal{P}| = \max_{i=1, \dots, n_{\mathcal{P}}} |t_i - t_{i-1}|$. The length of a path $\gamma : [a, b] \rightarrow M$ on a metric space (M, d) is given by

$$\ell_d(\gamma) = \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} d(\gamma(t_i), \gamma(t_{i-1})).$$

It is well known that for every $\varepsilon > 0$, there exist a $\delta > 0$ such that

$$\ell_d(\gamma) \leq \sum_{i=1}^{n_{\mathcal{P}}} d(\gamma(t_i), \gamma(t_{i-1})) + \varepsilon$$

for every partition \mathcal{P} such that $|\mathcal{P}| < \delta$ (see [6]).

Given a metric space (M, d) we can define a metric

$$\hat{d}(x, y) = \inf_{\gamma \in \mathcal{C}_{x,y}} \ell_d(\gamma),$$

on M , where $\mathcal{C}_{x,y}$ is the family of paths on (M, d) that connects x and y . We denote $\mathcal{C}_{x,y}^d$ instead of $\mathcal{C}_{x,y}$ if there exist more than one metric defined on M . The metric \hat{d} is the intrinsic metric induced by d , we always have that $d \leq \hat{d}$ and we say that d is intrinsic if $\hat{d} = d$ (see [6]).

If (M, d) is a metric space, $\varepsilon > 0$ and $x, y \in M$, then an ε -midpoint of x and y is a point $z \in M$ that satisfies $|2d(x, z) - d(x, y)| \leq \varepsilon$ and $|2d(y, z) - d(x, y)| \leq \varepsilon$.

We have the following results about intrinsic metrics. Their proofs can be found in [6].

Proposition 2.2. *If d is an intrinsic metric on M and $\varepsilon > 0$, then every $x, y \in M$ admit an ε -midpoint.*

Proposition 2.3. *Let (M, d) be a complete metric space. If every $x, y \in M$ admit a ε -midpoint (for every $\varepsilon > 0$), then d is intrinsic.*

Let M be a differentiable manifold and $TM = \{(x, v); x \in M, v \in T_x M\}$ be its tangent bundle. A C^0 -Finsler metric on M is a continuous function $F : TM \rightarrow \mathbb{R}$ such that $F(x, \cdot)$ is a norm on $T_x M$ for every $x \in M$. F induces a metric d_F on M given by

$$d_F(x, y) = \inf_{\gamma \in \mathcal{S}_{x,y}} \ell_F(\gamma),$$

where

$$\ell_F(\gamma) = \int_a^b F(\gamma'(t)) dt,$$

is the length of γ in (M, F) and $\mathcal{S}_{x,y}$ is the family of paths on M which are smooth by parts and connects x and y . We have that d_F is an intrinsic metric because

$$(1) \quad \hat{d}_F(x, y) = \inf_{\gamma \in \mathcal{C}_{x,y}} \ell_{d_F}(\gamma) \leq \inf_{\gamma \in \mathcal{S}_{x,y}} \int_a^b F(\gamma'(t)) dt = d_F(x, y) \leq \hat{d}_F(x, y).$$

If $d : M \times M \rightarrow \mathbb{R}$ is a metric on M , then we say that d is C^0 -Finsler if there exist a C^0 -Finsler metric F on M such that $d = d_F$.

Remark 2.4. *There are another usual (in fact more usual) definition of Finsler manifold, where F satisfies other conditions (see, for instance, [1] and [10]): F is smooth on $TM - TM_0$, where $TM_0 = \{(p, 0) \in TM; p \in M\}$, and $F(p, \cdot)$ is a Minkowski norm on $T_p M$ for every $p \in M$. In order to make this difference clear put the prefix C^0 before Finsler.*

C^0 -Finsler metrics are studied, for instance, in [3], [4] and [6].

Definition 2.5. *Let M be a set and consider two metrics d and ρ on M . We say that d and ρ are locally Lipschitz equivalent (see [2, 9]) if they induces the same topology on M and if for every $x \in M$, there exist an neighborhood V of x in (M, d) (or (M, ρ)) such that $d|_V$ and $\rho|_V$ are Lipschitz equivalent. If M is a differentiable manifold, we say that d is locally Lipschitz equivalent to a Finsler metric F if d is locally Lipschitz equivalent to d_F . Notice that this definition does not depend on the Finsler metric that is chosen because all Finsler metrics are locally Lipschitz equivalent.*

Theorem 2.6. *Let $\varphi : G \times M \rightarrow M$ be a transitive left action by isometries of a Lie group G on a metric space (M, d) . Suppose that d is intrinsic and locally Lipschitz equivalent to a Finsler metric. Then d is a left invariant Finsler metric.*

Proof

See Corollary 7.12 and Theorem 3.1 of [9] for a proof of this theorem.

This theorem can be also proved using Theorem 7 of [4], that states that (M, d) is isometric to a homogeneous Finsler manifold (M, F, d_F) iff

- The space (M, d) is locally compact, locally contractible homogeneous space with intrinsic metric;
- The orbits of the one-parameter group of motions in (M, d) are rectifiable.

The first condition obviously holds and it implies that M is isometric to a quotient space G/H of a Lie group G over a compact subgroup H of it, endowed with a G -invariant Carnot-Caratheodory-Finsler metric (see Theorem 3 in [3]). Then it is straightforward that if d is locally Lipschitz equivalent to a Finsler metric, then any path in (M, d) defined on a compact interval is rectifiable, and the result follows. ■

The rest of this section is about the orbits of a family of vector fields and it is used in Section 4 in order to prove the existence of X such that S_X is dense.[13] is its main reference.

Definition 2.7. *Let \mathcal{F} be a family of complete vector fields on a differentiable manifold M and $x \in M$. For $X \in \mathcal{F}$, denote by $t \mapsto (\exp tX)(x)$ be the integral curve of X such that $(\exp 0X)(x) = x$. The orbit $G(x)$ of \mathcal{F} through x is the set of points given by $(\exp t_m X_m)(\exp t_{m-1} X_{m-1}) \dots (\exp t_1 X_1)(x)$, with $m \in \mathbb{N}$, $t_i \in \mathbb{R}$ and $X_i \in \mathcal{F}$, $i = 1, \dots, m$.*

In the conditions of Definition 2.7, denote by $Lie(\mathcal{F})$ the Lie algebra generated by \mathcal{F} and let $Lie_x(\mathcal{F})$ be the restriction of $Lie(\mathcal{F})$ to the tangent space $T_x M$.

Theorem 2.8 (Hermann-Nagano Theorem). *Let M be an analytic manifold and \mathcal{F} be a family of analytic vector fields on M . Then*

- each orbit of \mathcal{F} is an analytic submanifold of M ;
- if N is an orbit of \mathcal{F} , then the tangent space of N in x is given by $\text{Lie}_x(\mathcal{F})$.

Corollary 2.9. *Let G be a connected Lie group and V be a vector subspace of \mathfrak{g} such that the Lie algebra generated by V is \mathfrak{g} . Let $\{v_1, \dots, v_k\}$ be a basis of V and \mathcal{F} be the set of left (or right) invariant vector fields with respect to $\{v_1, \dots, v_k\}$. Then, for every $x \in G$, there exist $m \in \mathbb{N}$, $t_{i_1}, \dots, t_{i_m} \in \mathbb{R}$ and $v_{i_1}, \dots, v_{i_m} \in V$ such that $x = \exp(t_{i_m} v_{i_m}) \dots \exp(t_{i_1} v_{i_1})$.*

Proof

It is enough to observe that G is an analytic manifold, that the right (and left) invariant vector fields with respect to v_1, \dots, v_k are analytic (see [12]) and that $\text{Lie}_e(\mathcal{F}) = \mathfrak{g}$. Then the orbit of e is an analytic submanifold of G that contains a neighborhood of e (see Theorem 2.8) and this neighborhood generates G due to the connectedness of G . ■

3. CONVEGENCE OF INDUCED HAUSDORFF METRICS IN LIE GROUPS

Let $\varphi : G \times (G, d) \rightarrow (G, d)$ be the product of G , d be a metric on G that is Lipschitz equivalent to a right invariant Riemannian metric on G and $X \ni e$ be a finite subset of G such that $H_X = \{e\}$. In this section, we study the sequence of metrics d, d^1, d^2, \dots on G .

First of all we prove that $\tau_{d_X} = \tau_d$ (Proposition 3.6).

Proposition 3.1. $d_M(p, q) := \max_{j=1, \dots, k} d(px_j, qx_j)$ is a metric on G .

Proof

Just observe that $(p, q) \mapsto d(px_j, qx_j)$ is a metric on G for every j and that the maximum of a finite family of metrics is also a metric. ■

Lemma 3.2. $d \leq d_M$ and $d_X \leq d_M$.

Proof

The first inequality is obvious. The second inequality follows because

$$\begin{aligned} d_X(p, q) &= \max \left\{ \max_i \min_j d(px_i, qx_j), \max_j \min_i d(px_i, qx_j) \right\} \\ &\leq \max \left\{ \max_i d(px_i, qx_i), \max_j d(px_j, qx_j) \right\} = \max_i d(px_i, qx_i) = d_M(p, q). \blacksquare \end{aligned}$$

Lemma 3.3. *For every $g \in G$, there exist a G -neighborhood V of $g \in G$ such that $d_X|_{V \times V} = d_M|_{V \times V}$.*

Proof

For $i, j = 1, \dots, k$, $i \neq j$, define $\rho_{ij} : G \times G \rightarrow \mathbb{R}$, as $\rho_{ij}(p, q) = d(px_i, qx_j) - \max_k d(px_k, qx_k)$. Then for every $g \in G$, there exist a G -neighborhood V of g such that

$$\rho_{ij}(p, q) > 0 \text{ for every } p, q \in V \text{ and every } i \neq j,$$

because $\rho_{ij}(g, g) > 0$. It implies that

$$\begin{aligned} d_X(p, q) &= \max \left\{ \max_i \min_j d(px_i, qx_j), \max_j \min_i d(px_i, qx_j) \right\} \\ (2) \quad &= \max \left\{ \max_i d(px_i, qx_i), \max_j d(px_j, qx_j) \right\} = \max_i d(px_i, qx_i) = d_M(p, q) \end{aligned}$$

for every $p, q \in V$. ■

Lemma 3.4. $\tau_d = \tau_{d_M} (= \tau_G)$ on G .

Proof

The inequality $d \leq d_M$ implies that $\tau_d \subset \tau_{d_M}$.

In order to see that $\tau_{d_M} \subset \tau_d$, consider an open ball $B_{d_M}(p, r)$. We prove that there exist an $\varepsilon > 0$ such that $B_d(p, \varepsilon) \subset B_{d_M}(p, r)$. For every $j = 1, \dots, k$, consider the open ball $B_j(p, r)$ with respect to the metric $(p, q) \mapsto d(px_j, qx_j)$. Observe that $q \mapsto qx_j$ is a homeomorphism and $B_j(p, r)$ is an open subset of G . Then

$$\bigcap_{j=1}^k B_j(p, r) \subset B_{d_M}(p, r)$$

is a G -neighborhood of $p \in G$ and this fact implies that there exist an open ball $B_d(p, \varepsilon) \subset \bigcap_{j=1}^k B_j(p, r) \subset B_{d_M}(p, r)$. ■

Lemma 3.5. $B_{d_X}(p, r)$ is contained in a compact subset of G .

Proof

Just observe that

$$B := \bigcup_{i=1}^k B_d(px_i, r) \supset B_{d_X}(p, r).$$

In fact, if $y \notin B$, then

$$\begin{aligned} d_X(p, y) &= \max \left\{ \max_i \min_j d(px_i, yx_j), \max_j \min_i d(px_i, yx_j) \right\} \\ &\geq \max \left\{ \max_i \min_j d(px_i, yx_j), \min_i d(px_i, y \cdot e) \right\} \\ &\geq \max \left\{ \max_i \min_j d(px_i, yx_j), r \right\} \geq r, \end{aligned}$$

where the second inequality is due to $y \notin B$. The result follows because B is contained in a G -compact subset of G .

Proposition 3.6. $\tau_d = \tau_{d_X}$ on G .

Proof

It is enough to prove that $\tau_{d_X} = \tau_{d_M}$ due to Lemma 3.4.

We know that $d_X \leq d_M$ (Lemma 3.2), what implies that $\tau_{d_X} \subset \tau_{d_M}$.

In order to prove that $\tau_{d_M} \subset \tau_{d_X}$, consider an open ball $B_{d_M}(p, r)$. We will find an $\varepsilon > 0$ such that $B_{d_X}(p, \varepsilon) \subset B_{d_M}(p, r)$. Consider a neighborhood V of p according to Lemma 3.3. We can eventually consider a smaller r in such a way that

$B_{d_M}(p, r) \subset V$ (see Lemma 3.4). Then $B_{d_X}(p, r) \cap V = B_{d_M}(p, r)$ due to Lemma 3.3.

If $B_{d_X}(p, r) - V$ is the empty subset, then we have that $B_{d_X}(p, r) \subset B_{d_M}(p, r)$ and we are done. Otherwise we consider

$$\varepsilon = \inf_{x \in B_{d_X}(p, r) - V} d_X(p, x).$$

Observe d_X is d -continuous because $\tau_{d_X} \subset \tau_d$ and ε is strictly positive because p is not contained in the compact subset $\overline{B_{d_X}(p, r) - V}$ (see Lemma 3.5). Therefore

$$B_{d_X}\left(p, \frac{\varepsilon}{2}\right) = B_{d_X}\left(p, \frac{\varepsilon}{2}\right) \cap V \subset B_{d_M}(p, r)$$

what settles the proposition. ■

Definition 3.7. Let $X = \{x_1, \dots, x_k\} \ni e$ be a finite subset of a Lie group G . The semigroup generated by X is defined as

$$S_X = \{x_{i_1} \dots x_{i_m}; m \in \mathbb{N}, i_j \in (1, \dots, k), j \in (1, \dots, m)\}.$$

In what follows, if $X = \{x_1, \dots, x_k\}$, then $X^{-1} := \{x_1^{-1}, \dots, x_k^{-1}\}$.

Proposition 3.8. Let X be a finite subset of a Lie group G . Then S_X is dense in G iff $S_{X^{-1}}$ is dense in G .

Proof

It is enough to observe that if $i : G \rightarrow G$ is the inversion map, then $i(S_X) = S_{X^{-1}}$. ■

Proposition 3.9. Let M be a set and suppose that d and ρ are Lipschitz equivalent metrics on M satisfying $cd \leq \rho \leq Cd$, $c, C > 0$. If d is complete, then ρ is also complete.

Proof

Let z_i be a Cauchy sequence in (M, ρ) . Then it is also a Cauchy sequence in (M, d) because $d(z_i, z_j) \leq (1/c)\rho(z_i, z_j)$ and we have that $z_i \rightarrow z$ in (M, d) . Therefore $z_i \rightarrow z$ in (M, ρ) because $\rho(z_i, z) \leq Cd(z_i, z)$. ■

Lemma 3.10. Let M be a differentiable manifold and suppose that d^i is a sequence of complete intrinsic metrics on M that converges pointwise to a complete metric d^∞ . Let \mathbf{g} be a complete Riemannian metric on M such that $cd_{\mathbf{g}} \leq d^i \leq Cd_{\mathbf{g}}$ for every i , where $c, C > 0$. Then d^∞ is intrinsic.

Proof

Let $\varepsilon > 0$ and $x, y \in M$. We will prove that x and y admits an ε -midpoint $z \in (M, d^\infty)$ (see Proposition 2.3).

Let z_i be an $\varepsilon/3$ -midpoint of (M, d_i) (see Proposition 2.2). We claim that the sequence z_i is contained in a compact subset. In fact

$$|2d^i(x, z_i) - d^i(x, y)| \leq \frac{\varepsilon}{3}$$

implies that

$$2d^i(x, z_i) \leq d^i(x, y) + \frac{\varepsilon}{3}$$

and

$$2cd_{\mathbf{g}}(x, z_i) \leq 2d^i(x, z_i) \leq d^i(x, y) + \frac{\varepsilon}{3} \leq Cd_{\mathbf{g}}(x, y) + \frac{\varepsilon}{3}.$$

Then the sequence z_i is contained in a bounded subset of a complete Riemannian manifold and consequently it is contained in a compact subset.

Let z be an accumulation point of z_i . We claim that z is an ε -midpoint of d^∞ . First of all, taking a subsequence z_i converging to z , we claim that $d^i(x, z_i)$ converges to $d^\infty(x, z)$ as i goes to infinity. In fact, for every $\epsilon > 0$, there exist $N_1 \in \mathbb{N}$ such that

$$\begin{aligned} |d^\infty(x, z) - d^i(x, z_i)| &\leq |d^\infty(x, z) - d^i(x, z) + d^i(x, z) - d^i(x, z_i)| \\ &\leq |d^\infty(x, z) - d^i(x, z)| + C|d_{\mathbf{g}}(x, z) - d_{\mathbf{g}}(x, z_i)| < \epsilon \end{aligned}$$

for every $i \geq N_1$, and $d^i(x, z_i)$ converges to $d^\infty(x, z)$. Then for every $\varepsilon > 0$, there exist a $N_2 \in \mathbb{N}$ such that

$$\begin{aligned} &|2d^\infty(x, z) - d^\infty(x, y)| \\ &\leq |2d^\infty(x, z) - 2d^i(x, z_i)| + |2d^i(x, z_i) - d^i(x, y)| + |d^i(x, y) - d^\infty(x, y)| \leq \varepsilon \end{aligned}$$

for every $i \geq N_2$. The inequality

$$|2d^\infty(y, z) - d^\infty(x, y)| \leq \varepsilon$$

is analogous. Then z is an ε -midpoint of x and y in (M, d^∞) and d^∞ is intrinsic due to Proposition 2.3. ■

Now we prove the main theorem of this work.

Theorem 3.11. *Let G be a Lie group, \mathbf{g} be a right invariant Riemannian metric on G . Let $d : G \times G \rightarrow \mathbb{R}$ be a metric on G which is Lipschitz equivalent to $d_{\mathbf{g}}$ and consider $c, C > 0$ such that $cd_{\mathbf{g}} \leq d \leq Cd_{\mathbf{g}}$. Let $\varphi : G \times (G, d) \rightarrow (G, d)$ be the product of G and $X = \{x_1, \dots, x_k\} \ni e$ be a finite subset of G such that $H_X = \{e\}$. Then*

- (1) $cd_{\mathbf{g}} \leq d^i \leq Cd_{\mathbf{g}}$ for every $i \in \mathbb{N}$;
- (2) $d \leq d^1 \leq d^2 \leq \dots \leq d^i \dots$;
- (3) d^i converges pointwise to a metric d^∞ , which is Lipschitz equivalent to $d_{\mathbf{g}}$;
- (4) d^∞ is intrinsic and $(d^\infty)^1 = d^\infty$.
- (5) $d^\infty(gs, hs) \leq d^\infty(g, h)$ if $s \in X$ and $d^\infty(gs, hs) \geq d^\infty(g, h)$ if $s^{-1} \in X$;
- (6) If S_X (or $S_{X^{-1}}$) is dense in G , then d^∞ is the distance function of a right invariant C^0 -Finsler metric on G .

Proof

Item (1).

Let $x, y \in G$ and $\gamma : [a, b] \rightarrow G$ be a d_X -path (which is also a G -path due to Proposition 3.6) connecting x and y . Then

$$\ell_{d_X}(\gamma) = \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} d_X(\gamma(t_i), \gamma(t_{i-1})).$$

Cover $\gamma([a, b])$ with open subsets V of Lemma 3.3 such that (2) holds. Let ε be the d -Lebesgue number of this covering. Let $\delta > 0$ such that if $|t_1 - t_2| < \delta$, then

$d(\gamma(t_2), \gamma(t_1)) < \varepsilon$. Consider only partitions with norm less than δ . Then

$$(3) \quad \ell_{d_X}(\gamma) = \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} \max_j d(\gamma(t_i)x_j, \gamma(t_{i-1})x_j)$$

due to (2). In one hand we have that

$$(4) \quad \begin{aligned} \ell_{d_X}(\gamma) &\leq C \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} \max_j d_{\mathbf{g}}(\gamma(t_i)x_j, \gamma(t_{i-1})x_j) \\ &= C \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} d_{\mathbf{g}}(\gamma(t_i), \gamma(t_{i-1})) = C \ell_{d_{\mathbf{g}}}(\gamma) \end{aligned}$$

because $d_{\mathbf{g}}$ is right invariant. On the other hand

$$(5) \quad \ell_{d_X}(\gamma) \geq \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} d(\gamma(t_i)e, \gamma(t_{i-1})e) = \ell_d(\gamma) \geq c \ell_{d_{\mathbf{g}}}(\gamma).$$

From (4) and (5) and the fact that $d_{\mathbf{g}}$ is intrinsic (see (1)), we have that

$$\begin{aligned} c d_{\mathbf{g}}(x, y) &= c \inf_{\gamma \in \mathcal{C}_{x,y}^{d_{\mathbf{g}}}} \ell_{d_{\mathbf{g}}}(\gamma) \leq \inf_{\gamma \in \mathcal{C}_{x,y}^{d_X}} \ell_{d_X}(\gamma) = d^1(x, y) \\ &\leq C \inf_{\gamma \in \mathcal{C}_{x,y}^{d_{\mathbf{g}}}} \ell_{d_{\mathbf{g}}}(\gamma) = C d_{\mathbf{g}}(x, y) \end{aligned}$$

for every $x, y \in G$ (Observe that $\mathcal{C}_{x,y}^{d_X} = \mathcal{C}_{x,y}^{d_{\mathbf{g}}}$ due to Proposition 3.6).

In order to get $c d_{\mathbf{g}} \leq d^i \leq C d_{\mathbf{g}}$, it is enough to replace d by d^{i-1} .

Item (2).

Notice that a

$$d^1(x, y) = \inf_{\gamma \in \mathcal{C}_{x,y}^{d_X}} \ell_{d_X}(\gamma) \geq \inf_{\gamma \in \mathcal{C}_{x,y}^d} \ell_d(\gamma) = \hat{d}(x, y) \geq d(x, y)$$

due to (5). We settle Item (2) replacing d by d^{i-1} .

Item (3).

Observe that d^i is an increasing sequence of metrics bounded above by $C d_{\mathbf{g}}$. Then d^i converges pointwise to a function d^∞ , and it is easy to prove that d^∞ is a metric. The Lipschitz equivalence between d^∞ and $d_{\mathbf{g}}$ is also straightforward. This settles Item (3).

Item (4).

d^∞ is intrinsic due to Lemma 3.10. Let us prove that $(d^\infty)^1 = d^\infty$. For every $\varepsilon > 0$, there exist a $\delta > 0$ such that if $|\mathcal{P}| < \delta$, then

$$(6) \quad \ell_{d_X^\infty}(\gamma) \leq \sum_{i=1}^{n_{\mathcal{P}}} d_X^\infty(\gamma(t_i), \gamma(t_{i-1})) + \frac{\varepsilon}{2}.$$

We can consider δ smaller than the δ of Item (1), with d replaced by d^∞ . We have that

$$\ell_{d_X^\infty}(\gamma) \leq \sum_{i=1}^{n_P} \max_j d^\infty(\gamma(t_i)x_j, \gamma(t_{i-1})x_j) + \frac{\varepsilon}{2}$$

due to (2) and (6). For such a fixed \mathcal{P} , there exist a sufficiently large k such that

$$\ell_{d_X^\infty}(\gamma) \leq \sum_{i=1}^{n_P} \max_j d^k(\gamma(t_i)x_j, \gamma(t_{i-1})x_j) + \varepsilon \leq \ell_{d_X^k}(\gamma) + \varepsilon \leq \ell_{d^\infty}(\gamma) + \varepsilon,$$

what implies that

$$(7) \quad \ell_{d_X^\infty}(\gamma) \leq \ell_{d^\infty}(\gamma).$$

Then

$$(d^\infty)^1(x, y) = \inf_{\gamma \in \mathcal{C}_{x,y}^{d_X^\infty}} \ell_{d_X^\infty}(\gamma) \leq \inf_{\gamma \in \mathcal{C}_{x,y}^{d^\infty}} \ell_{d^\infty}(\gamma) \leq d^\infty(x, y) \leq (d^\infty)^1(x, y).$$

The second inequality holds because d^∞ is intrinsic. The last inequality holds due to Item (2). Consequently $(d^\infty)^1 = d^\infty$.

Item (5).

Notice that

$$\ell_{d^\infty}(\gamma) \geq \sup_{\mathcal{P}} \sum_{i=1}^{n_P} \max_j d^\infty(\gamma(t_i)x_j, \gamma(t_{i-1})x_j)$$

due to Equation (7). Then

$$\ell_{d^\infty}(\gamma) \geq \sup_{\mathcal{P}} \sum_{i=1}^{n_P} d^\infty(\gamma(t_i)x_j, \gamma(t_{i-1})x_j) = \ell_{d^\infty}(\gamma x_j)$$

for every $j = 1, \dots, k$. This is enough to prove that

$$(8) \quad d^\infty(g, h) \geq d^\infty(gx_j, hx_j)$$

for every $j = 1, \dots, k$. The inequality $d^\infty(gx_j^{-1}, hx_j^{-1}) \geq d^\infty(g, h)$ holds replacing g and h by gx_j^{-1} and hx_j^{-1} respectively in (8).

Item (6).

Observe that $d^\infty(gs, hs) \leq d^\infty(g, h)$ holds for every $s \in S_X$ and $d^\infty(gs, hs) \geq d^\infty(g, h)$ holds for every $s \in S_{X^{-1}}$ due to Item (5). Notice that $d^\infty(gu, hu) = d^\infty(g, h)$ for every $g, h, u \in G$. In fact, the inequality $d^\infty(gu, hu) \leq d^\infty(g, h)$ holds because there exist a sequence $s_i \rightarrow u$ in S_X and the inequality $d^\infty(gu, hu) \geq d^\infty(g, h)$ holds because there exist a sequence $s_i \rightarrow u$ in $S_{X^{-1}}$. Then d^∞ is a right invariant metric on G .

In order to see that d^∞ is a distance function of an right invariant Finsler metric, observe that if we consider the action $\phi : G \times G \rightarrow G$ defined as $\phi(g, h) = \varphi(g^{-1}, h)$, then ϕ is a transitive action of G on G and d^∞ is a ϕ -left-invariant metric on G (see Remark 2.1). Then we are in the conditions of Theorem 2.6 and d^∞ is a ϕ -invariant C^0 -Finsler metric, that is, a right invariant C^0 -Finsler metric. ■

4. EXISTENCE OF DENSE SEMIGROUPS S_X

Let G be a connected Lie group. In this section we prove the existence of a finite subset $X = \{x_1, \dots, x_k\} \ni e$ of G such that $H_X = \{e\}$ and $\overline{S_X} = G$.

Example 4.1. Let $G = \mathbb{R}$ be the additive group of real numbers and consider $X = \{-1, 0, \sqrt{2}\}$. We claim that S_X is dense. Let $q \in \mathbb{R}$ and $\varepsilon > 0$. We will find a $q_\varepsilon \in S_X$ such that $|q - q_\varepsilon| < \varepsilon$.

It is easy to see that there are a sequence p_1, p_2, \dots such that $p_{i+1} - p_i = -1$ or $p_{i+1} - p_i = \sqrt{2}$ and a infinite number of p_i 's are in $[0, 1]$. Therefore there exist p_j and p_k in S_X , $j < k$, such that $|p_k - p_j| < \varepsilon$. Set $p_\varepsilon := p_k - p_j \in S_X$. In order to fix ideas, suppose that $p_\varepsilon > 0$. Then there exist an element $p \in S_X$ such that $p < q$ and a $l \in \mathbb{N}$ such that $q_\varepsilon := p + l p_\varepsilon$ satisfies $|q - q_\varepsilon| < \varepsilon$. The case $p_\varepsilon < 0$ is analogous. Then S_X is dense in \mathbb{R} . We also have that $S_{X^{-1}}$ is dense in G due to Proposition 3.8.

This example is easily generalized to the case $G = \mathbb{R}^n$ and $X = \{-1, 0, \sqrt{2}\} \times \dots \times \{-1, 0, \sqrt{2}\}$. ■

Example 4.1 can be generalized to Lie groups in the following way:

Theorem 4.2. Let G be a connected Lie group and \mathfrak{g} be its Lie algebra. Let V be a vector subspace of \mathfrak{g} such that the Lie algebra generated by V is \mathfrak{g} . Let $\{v_1, \dots, v_k\}$ be a basis of V such that

$$\left\{ \sum_{i=1}^k a_i v_i, a_i \in [-2, 2], i = 1, \dots, k \right\}$$

is contained in a neighborhood A of $0 \in \mathfrak{g}$ such that $\exp|_A$ is a diffeomorphism over its image. Consider

$$\tilde{X} = \{-v_1, -v_2, \dots, -v_k, 0, \sqrt{2}v_1, \dots, \sqrt{2}v_k\}.$$

Denote $x_i = \exp(-v_i)$ and $y_i = \exp(\sqrt{2}v_i)$ and $X = \{e, x_1, \dots, x_k, y_1, \dots, y_k\}$. Then $H_X = \{e\}$ and S_X is dense in G .

Proof

Let $g \in H_X$. Then $gX = X$ and $e \in X$ implies that $g \in X$ and $g^{-1} \in X$. If $g \neq e$, then $g = \exp(-v_i)$ or $g = \exp(\sqrt{2}v_i)$. In order to fix ideas, suppose that $g = \exp(-v_i)$. Then we have that $g^{-1} = \exp(v_i) \in X$, which isn't possible because $\exp|_A$ is a diffeomorphism over its image. The other case is analogous. Consequently $H_X = \{e\}$.

Let us prove that S_X is dense in G . Example 4.1 states that $S_{\tilde{X}} \cap \mathbb{R}v_i$ is dense in $\mathbb{R}v_i$ for every $i = 1, \dots, n$. This implies that $S_X \cap \exp(\mathbb{R}v_i)$ is dense in $\exp(\mathbb{R}v_i)$. By Corollary 2.9, for every $x \in G$, there exist $m \in \mathbb{N}$, $t_{i_1}, \dots, t_{i_m} \in \mathbb{R}$ and $v_{i_1}, \dots, v_{i_m} \in X$ such that $x = \exp(t_{i_m} v_{i_m}) \dots \exp(t_{i_1} v_{i_1})$. But we know that for every $j = 1, \dots, m$, there exist a sequence of points in S_X converging to $\exp(t_{i_j} v_{i_j})$. Therefore we have a sequence of points in S_X converging to x and S_X is dense in G . ■

5. CONCLUSIONS

From a Lie group G endowed with a metric which is only Lipschitz equivalent to a right invariant Riemannian metric, we defined a discrete dynamical system d, d^1, \dots that converges to a right invariant C^0 -Finsler metric. This is the first

example that shows that a sequence of metrics defined from the Hausdorff distance can have a regularizing effect. We intend to study the properties of this sequence in other situations.

REFERENCES

- [1] D. Bao, S.-S. Chern, and Z. Shen, *An introduction to Riemann-Finsler geometry*, Graduate Texts in Mathematics, vol. 200, Springer-Verlag, New York, 2000. MR1747675
- [2] Djeison Benetti, *Induced Hausdorff metrics on homogeneous spaces*, Ph.D. Thesis, State University of Maringá, Brazil, 2016. In Portuguese.
- [3] V. N. Berestovskii, *Homogeneous manifolds with an intrinsic metric. I*, Sibirsk. Mat. Zh. **29** (1988), no. 6, 17–29, DOI 10.1007/BF00972413 (Russian); English transl., Siberian Math. J. **29** (1988), no. 6, 887–897 (1989). MR985283
- [4] ———, *Homogeneous manifolds with an intrinsic metric. II*, Sibirsk. Mat. Zh. **30** (1989), no. 2, 14–28, 225, DOI 10.1007/BF00971372 (Russian); English transl., Siberian Math. J. **30** (1989), no. 2, 180–191. MR997464
- [5] Arthur L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition. MR2371700
- [6] Dmitri Burago, Yuri Burago, and Sergei Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418
- [7] Wei-Liang Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann. **117** (1939), 98–105 (German). MR0001880
- [8] Bennett Chow and Dan Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, Providence, RI, 2004. MR2061425
- [9] Ryuichi Fukuoka and Djeison Benetti, *Induced Hausdorff metrics on quotient spaces*, arXiv:1604.07129, to appear in Bull. Braz. Math. Soc. (N.S.) (2016).
- [10] Shaoqiang Deng, *Homogeneous Finsler spaces*, Springer Monographs in Mathematics, Springer, New York, 2012. MR2962626
- [11] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. MR664497
- [12] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original. MR1834454
- [13] Velimir Jurdjevic, *Geometric control theory*, Cambridge Studies in Advanced Mathematics, vol. 52, Cambridge University Press, Cambridge, 1997. MR1425878
- [14] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original; A Wiley-Interscience Publication. MR1393940
- [15] ———, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996. Reprint of the 1969 original; A Wiley-Interscience Publication. MR1393941
- [16] John Mitchell, *On Carnot-Carathéodory metrics*, J. Differential Geom. **21** (1985), no. 1, 35–45. MR806700
- [17] Richard Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002. MR1867362
- [18] John Morgan and Gang Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.
- [19] P. K. Rashevskii, *About connecting two points of complete nonholonomic space by admissible curve*, Uch. Zap. ped. inst. Libknexa **2**, 83–94 (Russian).
- [20] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition. MR722297

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